FREE CONVECTIVE HEAT TRANSFER AT A VERTICAL SEMIINFINITE PLATE

A. A. Berezovskii, O. G. Martynenko,
and Yu. A. Sokovishin

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The effect of the boundary layer at the leading edge on heat transfer near a vertical semiinfinite heated plate is determined by means of matched asymptotic expansions. The criterial relation for air is in good agreement with existing experimental data.

The study of laminar free convection in a medium adjacent to the outer surface of a heat-emitting surface is a complex problem because the velocity and temperature fields have a mutual effect on one another. Until recently, an analytic study of free convection was only possible within the confines of boundary-layer theory [1, 2]. The use of perturbation methods, which have been actively developed in recent years, provides an opportunity to carry out a more thorough analysis of free convective heat transfer based on the use of the complete equations of motion and heat transfer and on consideration of the mutual effect of the boundary layer and the external flow induced by it. From the viewpoint of the method of matched asymptotic expansions [3, 4], the solution obtained from boundary-layer theory is an approximation of zeroth order in the asymptotic solution as $Gr \to \infty$ and is a good description of free convective heat transfer for very large values of the Grashof number. A correction must be made for moderate values of the Grashof number. The need for such a correction follows not only from mathematical considerations, but also from comparison with experiment [1]. The first correction of this kind was obtained in [5], where the approximation of first order was investigated for the problem of free convection near a finite vertical plate. However, the lack of information about the behavior of the solution in the region outside the plate and the deficiencies of the method used led to a correction to the mean coefficient of heat transfer that was negative, contrary to expectations. Explaining this by the impossibility of including the effects of the leading edge, the authors corrected the solution obtained by means of a study of free convective heat transfer at small Grashof numbers [6], which was confirmed in turn by direct numerical calculations [7], but they did not succeed in obtaining a unified pattern of flow and heat transfer.

A more successful study was one by the method of matched asymptotic expansions for the first three approximations in the problem of free convection near a vertical semiinfinite plate [8]. In particular, the expression

$$\frac{\mathrm{Nu}_x}{\mathrm{Gr}_x^{1/4}} = 0.3568 + 0.3877\mathrm{Gr}_x^{-1/2} + O(\mathrm{Gr}_x^{-3/4}) \tag{1}$$

(for Pr = 0.72) was obtained for the local coefficient of heat transfer.

The method of matched asymptotic expansions was used most systematically in [9]. The inner expansion was supplemented by eigensolutions, and the expression (for Pr = 0.72)

$$\frac{\mathrm{Nu}_x}{\mathrm{Gr}_x^{1/4}} = 0.3568 - 0.0702c_1\mathrm{Gr}_x^{-1/3} + 0.8915\mathrm{Gr}_x^{-1/2} + O(\mathrm{Gr}_x^{-3/4}) \tag{2}$$

was obtained as the result for the local coefficient of heat transfer, and the expression

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$$\frac{\mathrm{Nu}_L}{\mathrm{Gr}_L^{1/4}} = 0.476 + 0.625\mathrm{Gr}_L^{-1/4} + O(\mathrm{Gr}_L^{-1/3}) \tag{3}$$

for the mean coefficient of heat transfer.

The value of Eq. (2) is somewhat reduced because of the presence of the undetermined constant c_1 in the term corresponding to the eigensolution.

An analysis of the application of the method of matched asymptotic expansions to the problem of free convection near a vertical plate [5, 8, 9] shows that the first approximation has no effect on heat transfer; the inner solution has a self-similar nature for each approximation, which ensures matching; the solution is not applicable in the neighborhood of the leading edge. This is a consequence of a more general statement: A problem containing a certain parameter admits of a self-similar solution expressed through this parameter which cannot be uniformly valid unless it is an exact solution [4]. In addition, it is assumed in the solution that motion occurs only in the half-plane $x > 0$, which does not correspond to experimental observations.

The purpose of this paper is the development of a perturbation method based on the method of matched asymptotic expansions and deformed coordinates [3, 4] in order to include some effects of the leading edge. This is accomplished by a shift of the singularities in the solution in the direction of their true position by means of a slight deformation of the longitudinal coordinate through the expression

$$x = X + \varepsilon f(X, Y). \tag{4}$$

The form of the function f is determined in the course of solution from the matching conditions and the requirement for conservation of the self-similar nature of the solution.

The basic equations describing stationary laminar free convection near a vertical semi-infinite plate in the Boussinesq approximation written in terms of dimensionless stream function and temperature are [8]

$$\begin{gathered}\frac{\partial\psi}{\partial y}\cdot\frac{\partial}{\partial x}(\nabla^2\psi) - \frac{\partial\psi}{\partial x}\cdot\frac{\partial}{\partial y}(\nabla^2\psi) = \frac{1}{\mathrm{Gr}^{1/2}}\nabla^4\psi + \frac{\partial\theta}{\partial y},\\ \frac{\partial\psi}{\partial y}\cdot\frac{\partial\theta}{\partial x} - \frac{\partial\psi}{\partial x}\cdot\frac{\partial\theta}{\partial y} = \frac{1}{\mathrm{Pr}\mathrm{Gr}^{1/2}}\nabla^2\theta\end{gathered} \tag{5}$$

with the boundary conditions

$$\begin{gathered}\frac{\partial\psi}{\partial x} = \frac{\partial\psi}{\partial y} = 0,\ \theta = 1,\ y = 0,\ x > 0,\\ \frac{\partial^2\psi}{\partial y^2} = \frac{\partial\psi}{\partial x} = \frac{\partial\theta}{\partial y} = 0,\ y = 0,\ x < 0,\\ \frac{\partial\psi}{\partial x} = \frac{\partial\psi}{\partial y} = \theta \to 0,\ r \to \infty,\ \varphi \neq 0.\end{gathered} \tag{6}$$

The solution of the problem (5), (6) is represented in the form of asymptotic expansions in the boundary layer and in the outer flow:

outer expansion

$$\begin{gathered}\psi(x, y;\ \mathrm{Gr}) = \varepsilon\psi_1(x, y) + \varepsilon^2\psi_2(x, y) + \cdots,\\ \theta(x, y;\ \mathrm{Gr}) \equiv 0\end{gathered} \tag{7}$$

as $\varepsilon \to 0$, for fixed x, y;

inner expansion

$$\begin{gathered}\psi(x, y; \mathrm{Gr}) = \varepsilon\Psi_0(X, Y) + \varepsilon^2\Psi_1(X, Y) + \cdots,\\ \theta(x, y;\ \mathrm{Gr}) = \Theta_0(X, Y) + \varepsilon\Theta_1(X, Y) + \cdots\end{gathered} \tag{8}$$

as $\varepsilon \to 0$, for fixed X, Y.

The form of the expansions (7) and (8) can be made more general by determining the terms of the corresponding asymptotic sequences in each step of the solution from matching, but the result will be exactly the same, at least to second approximation inclusively. One can also write an outer expansion for θ, but the system (5) indicates that the temperature will be conserved along a streamline as $Gr \to \infty$; the temperature at infinity is constant and, consequently, the excess temperature is zero.

The form of the limiting process restricts the region of applicability of each of the expansions. Substitution of the expansions (7) and (8) into the system (5) yields a set of problems for determination of the coefficients in the asymptotic expansions. Boundary conditions in each step are determined from Eqs. (6) and matching.

For the zeroth approximation to the internal solution, we obtain the well-known problem in the theory of a thermal boundary layer

$$F_0''' + \frac{3}{4} F_0''F_0 - \frac{1}{2} F_0'F_0' + H_0 = 0, \tag{9}$$

$$H_0'' + \frac{3}{4} \Pr F_0 H_0' = 0,$$

$$F_0(0) = F_0'(0) = 0, \; H_0(0) = 1, \tag{10}$$

$$F_0'(\infty) = H_0(\infty) = 0,$$

where

$$\Psi_0 = X^{3/4} F_0(\eta), \; \Theta_0 = H_0(\eta), \; \eta = YX^{-1/4}. \tag{11}$$

Assuming the outer flow to be irrotational, we obtain for the determination of the first approximation to the outer solution

$$\nabla^2 \psi_1 = 0, \tag{12}$$

$$\psi_1(x, 0) = x^{3/4} F_0(\infty) \quad (x > 0),$$

$$\psi_1(x, 0) = 0 \quad (x < 0). \tag{13}$$

The solution of the problem (12), (13) is given by the Poisson integral for a half-plane and has the form

$$\psi_1(x, y) = -\sqrt{2} F_0(\infty) r^{3/4} \sin \frac{3}{4} (\varphi - \pi). \tag{14}$$

Matching of the inner and outer expansions determines the boundary condition for the first approximation to the inner solution:

$$\frac{\partial \Psi_1}{\partial Y}(X, \infty) = \frac{3}{4} F_0(\infty) X^{-1/4} + f_Y \frac{3}{4} F_0(\infty) X^{-1/4}. \tag{15}$$

In accordance with the condition for exponential decrease in vorticity, we assume

$$\frac{\partial \Psi_1}{\partial Y}(X, \infty) = 0; \tag{16}$$

consequently,

$$f_Y' = -1. \tag{17}$$

For determination of the first approximation for the boundary layer, we have the self-similar problem

$$F_1''' + \frac{3}{4} F_1'' F_0 - \frac{1}{4} F_1'F_0' + H_1 = 0,$$

$$H_1'' + \frac{3}{4} \Pr (F_0 H_1)' = 0, \tag{18}$$

$$F_1(0)=F_1'(0)=H_1(0)=H_1(\infty)=0,$$
$$F_1'(\infty)=\frac{3}{4}F_0(\infty), \tag{19}$$

if one assumes

$$F_1(\eta)=\Psi_1-f\frac{\partial\Psi_0}{\partial X},$$
$$H_1(\eta)=X^{3/4}\left(\Theta_1-f\frac{\partial\Theta_0}{\partial X}\right). \tag{20}$$

From Eq. (17) and the condition for self-similarity (20), we have

$$f(X, Y)=-Y-kX^{1/4}. \tag{21}$$

The problem (18), (19) is solved numerically. Even without solution, however, one can determine that

$$H_1(\eta)=0, \tag{22}$$

and as $\eta\to\infty$

$$\Psi_1(\eta)\sim F_1(\infty)-\frac{3}{4}kF_0(\infty)+\exp, \tag{23}$$

where exp denotes terms which are exponentially small as $\eta\to\infty$. In view of Eq. (16), we assume

$$F_1(\infty)-\frac{3}{4}kF_0(\infty)=2F_1(\infty), \tag{24}$$

and then

$$k=-\frac{4}{3}\frac{F_1(\infty)}{F_0(\infty)}. \tag{25}$$

Thus, we finally have

$$f(X, Y)=-Y+\frac{4}{3}\frac{F_1(\infty)}{F_0(\infty)}X^{1/4}. \tag{26}$$

Equations (4) and (26) show that when $\bar{x}=0$ and $\bar{y}=0$, $X\sim 1/L$; i.e., a flow beginning somewhat below the leading edge is described by the proposed method. Since $\Theta_1\neq 0$, a correction to the boundary-layer theory appears even in the first approximation.

By deforming the outer longitudinal coordinate in a similar fashion,

$$x=X+\frac{4}{3}\frac{F_1(\infty)}{F_0(\infty)}X^{1/4}\varepsilon, \tag{27}$$

we obtain for the second approximation to the outer flow

$$\frac{\partial^2}{\partial X^2}\left(\psi_2-\frac{4}{3}\frac{F_1(\infty)}{F_0(\infty)}X^{1/4}\frac{\partial\psi_1}{\partial X}\right)+$$
$$+\frac{\partial^2}{\partial y^2}\left(\psi_2-\frac{4}{3}\frac{F_1(\infty)}{F_0(\infty)}X^{1/4}\frac{\partial\psi_1}{\partial X}\right)=0, \tag{28}$$

$$\psi_2(X, 0) = F_1(\infty) \quad (X > 0),$$
$$\psi_2(X, 0) = 0 \quad (X < 0). \tag{29}$$

The function

$$\psi_2(X, y) = \frac{4}{3}\frac{F_1(\infty)}{F_0(\infty)} X^{1/4}\frac{\partial\psi_1}{\partial X} + F_1(\infty)\left(1 - \frac{\varphi}{\pi}\right) \tag{30}$$

is a solution of the problem (28), (29). From matching we obtain boundary conditions for the second approximation to the boundary layer,

$$\frac{\partial\Psi_2}{\partial Y}(X, \infty) = \frac{\partial\psi_2}{\partial y}(x, 0) + ff_Y\frac{\partial^2\Psi_0}{\partial X^2} + f_Y\frac{\partial F_1}{\partial X} = -\frac{1}{\pi X}F_1(\infty), \tag{31}$$

$$\frac{\partial^2\Psi_2}{\partial Y^2}(X, \infty) = \frac{\partial^2\psi_1}{\partial y^2}(x, 0) + (f_Y)^2\frac{\partial^2\Psi_0}{\partial X^2} = 0. \tag{32}$$

In order to determine the second approximation to the boundary layer, we obtain the self-similar problem

$$F_2''' + \frac{3}{4}F_2''F_0 + \frac{1}{2}F_2'F_0' - \frac{3}{4}F_2F_0'' + H_2 = -\frac{1}{2}P_2 - \frac{1}{4}\eta P_2' -$$
$$-\frac{1}{4}F_1'F_1' + \frac{1}{4}F_0' - \frac{1}{16}\eta F_0'' - \frac{1}{16}\eta^2 F_0''', \tag{33}$$
$$H_2'' + \Pr\left(\frac{3}{4}F_0H_2' + \frac{3}{2}F_0'H_2 - \frac{3}{4}H_0'F_2\right) = -\frac{5}{16}\eta H_0' - \frac{1}{16}\eta^2 H_0'',$$

where

$$P_2' = \frac{1}{16}\eta F_0'F_0' + \frac{3}{16}\eta F_0F_0'' - \frac{9}{16}F_0F_0' + \frac{1}{4}\eta F_0''' - \frac{1}{4}F_0'', \tag{34}$$

if one assumes

$$F_2(\eta) = X^{3/4}\left(\Psi_2 - \frac{f^2}{2}\cdot\frac{\partial^2\Psi_0}{\partial X^2} - f\,\frac{\partial\Psi_1}{\partial X}\right),$$
$$H_2(\eta) = X^{3/2}\left(\Theta_2 - \frac{f^2}{2}\cdot\frac{\partial^2\Theta_0}{\partial X^2}\right). \tag{35}$$

The boundary conditions have the form

$$F_2(0) = F_2'(0) = H_2(0) = H_2(\infty) = 0,$$
$$F_2'(\infty) = -\frac{F_1(\infty)}{\pi}, \quad F_2''(\infty) = \frac{3}{16}F_0(\infty). \tag{36}$$

The problem (33), (36) is solved numerically. As $\eta \to \infty$

$$\Psi_2 \sim X^{3/4}\left(-\frac{F_1(\infty)}{\pi}\eta - \frac{8F_1^2(\infty)}{3F_0(\infty)} + F_2(\infty)\right) + \exp, \tag{37}$$

which ensures an exponential decrease in vorticity.

In order to find the eigensolutions, which can be omitted, we add to the inner expansion terms of the form

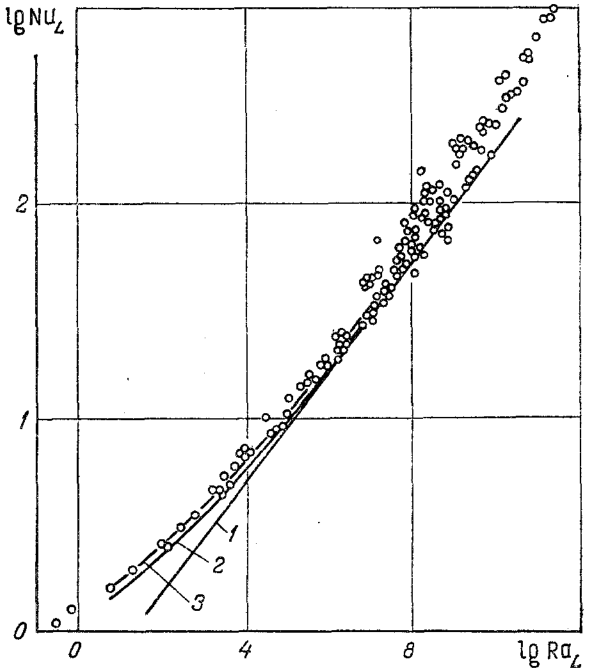

Fig. 1. Comparison of calculated and experimental results: 1) from boundary-layer theory; 2) from Eq. (3); 3) from Eq. (46).

$$c_k \varepsilon^{\alpha_k+1} \frac{f_k(\eta)}{X^{3/4(\alpha_k-1)}}, \quad c_k \varepsilon^{\alpha_k} \frac{g_k(\eta)}{X^{3/4\alpha_k}}. \tag{38}$$

For the determination of f_k and g_k we have

$$f_k''' + \frac{3}{4} f_k'' F_0 + \left(\frac{3}{4}\alpha_k - 1\right) f_k' F_0 + \frac{3}{4}(1-\alpha_k) f_k F_0'' + g_k = 0, \tag{39}$$

$$g_k'' + \frac{3}{4} \Pr (g_k' F_0 + \alpha_k g_k F_0' + (1-\alpha_k) f_k H_0') = 0,$$

$$f_k(0) = f_k'(0) = g_k(0) = f_k'(\infty) = g_k(\infty) = 0. \tag{40}$$

The first eigensolution, corresponding to the eigenvalue $\alpha = 4/3$, has the form [9]

$$f_1 = c_1 X^{1/4} \frac{\partial \Psi_0}{\partial X}, \quad g_1 = c_1 X^{3/4} \frac{\partial \Theta_0}{\partial X}. \tag{41}$$

Since the eigensolutions correspond to indeterminacy in the flow near the leading edge [3], and the solution for the first approximation to the boundary layer, or at least that portion of it which has the form

$$\frac{4}{3} \frac{F_1(\infty)}{F_0(\infty)} X^{1/4} \frac{\partial \Psi_0}{\partial X}, \quad \frac{4}{3} \frac{F_1(\infty)}{F_0(\infty)} X^{1/4} \frac{\partial \Theta_0}{\partial X}, \tag{42}$$

arises because of attempts to eliminate this indeterminacy, it is natural to set up the definition

$$c_1 = \frac{4}{3} \frac{F_1(\infty)}{F_0(\infty)}. \tag{43}$$

Defining the local coefficient of heat transfer by the expression

$$\alpha_{\bar{x}} (T_w - T_\infty) = -\lambda \left(\frac{\partial T}{\partial \bar{y}}\right)_{\bar{y}=0}, \tag{44}$$

we obtain for the local dimensionless coefficient of heat transfer

$$\frac{\mathrm{Nu}_x}{\mathrm{Gr}_x^{1/4}} = -H_0'(0) + \frac{F_1(\infty)}{3F_0(\infty)} \cdot \frac{H_0'(0)}{\mathrm{Gr}_x^{1/4}} + \frac{F_1(\infty)}{3F_0(\infty)} \cdot \frac{H_0'(0)}{\mathrm{Gr}_x^{1/3}} - $$

$$- \frac{5F_1^2(\infty)}{18F_0^2(\infty)} \cdot \frac{H_0'(0)}{\mathrm{Gr}_x^{1/2}} - \frac{H_2'(0)}{\mathrm{Gr}_x^{1/2}} + O(\mathrm{Gr}_x^{-3/4}). \quad (45)$$

A comparison of Eqs. (1), (2), and (45) shows that the latter has a more complicated structure resulting from the appearance of corrections because of the displacement of the solution, with the corrections being expressed through the local coefficient of heat transfer defined in boundary-layer theory. The effect of the Prandtl number appears implicitly in the values of $H_0'(0)$, $F_0(\infty)$, $F_1(\infty)$, and $H_2'(0)$.

Integration of Eq. (45) along the plate, which is possible because $X \neq 0$ at the leading edge, yields an expression for the mean coefficient of heat transfer (for Pr = 0.72):

$$\mathrm{Nu}_L = 0.476\,\mathrm{Gr}_L^{1/4} + 0.238 \ln \mathrm{Gr}_L^{1/4} + 0.605 + O(\mathrm{Gr}_L^{-1/12}). \quad (46)$$

A comparison of experimental and theoretical results for air is shown in Fig. 1. The proposed method yields good agreement of results in the region of moderate values of Gr. It is recommended that Eq. (46) not be used for $\mathrm{Gr}_L < 10$, since the error approaches the order of unity in this case. Equation (46) can be extended to other values of Pr also.

NOTATION

$\bar{x}$, $\bar{y}$, dimensional Cartesian coordinates; x, y, dimensionless coordinates; L, characteristic linear dimension; $\bar{\psi}$, stream function; T, temperature; $\theta = (T - T_\infty)/(T_w - T_\infty)$, dimensionless excess temperature; g, acceleration of gravity; β, coefficient of volumetric expansion; ν, coefficient of kinematic viscosity; a, coefficient of thermal diffusivity; λ, coefficient of thermal conductivity; α, coefficient of heat transfer; $\varepsilon = \mathrm{Gr}^{-1/4}$, expansion parameter; $\psi = \bar{\psi}(g\beta(T_w - T_\infty)L^3)^{1/2}$, dimensionless stream function; $Y = y/\varepsilon$, variable for internal expansion; $\eta = Y/X^{1/4}$, self-similar variable for internal expansion; $r = (\underline{x}^2 + y^2)^{1/2}$, $\varphi = \arctan y/x$, variables for external flow; $\mathrm{Pr} = \nu/a$, Prandtl number; $\mathrm{Nu}_x = \alpha_x \bar{x}/\lambda$, local Nusselt number; $\mathrm{Gr}_x = g\beta(T_w - T\ )\bar{x}^3/\nu^2$, local Grashof number. Indices: x, local values; ∞, external flow; w, wall; L, value averaged over plate length L.

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